Question

You are the Monarch of an Island and are about to have a festival tomorrow. The festival is the most important one you have ever had.

You've got 1000 bottles of wine you were planning to open for the fiesta, but you find out that one of them is poisoned by some evil scoundrel. The actual poison exhibits no symptoms until somewhere around the 23rd hour, then results in sudden death. You have thousands of prisoners at your disposal.

1. What is the smallest number of prisoners you must have to drink from the bottles to find the poisoned bottle?
2. What is the smallest number of prisoners that you must sacrifice?
3. For the strong minded, what is the smallest number of prisoners if more than one bottle (say 2) is poisoned?

Philosophy

The reason I think this problem is interesting to solve is because the way you approach the solution is a strong indication of how you visualize the problem. As with most problems in math and life, the difficulty of the solution is directly proportional to how and how well you understand the problem. Math is about getting rid of the fluff and getting down to the real crux of the matter. How well you are able to rid the problem of the fluff determines how easily you can solve it. There are several observations that you can make to get to the core of the question. I will try to point out the ones I used in this process. I must admit, my solution is NOT the most elegant, but that is because I could not get rid of all the fluff in the question. I still retained some notion of bottles and prisoners. I found a solution on the web which I thought was very elegant. That is the kind of solutions you get when you get rid of all the notions of time, space, emotions and characters from the problem. That I will present at the very end. I will give you hints in the next section, my solution in the following section. I have not yet solved the “more than 1 poisoned bottle” case and any help will be appreciated.

Hints

Read this section after you have given the problem a good think.
There is information provided in the problem which has to be seen for what it really is saying. That is part of the getting rid of the fluff. This method of understanding what is really behind the verbosity of the problem is very useful.

What is really behind the verbosity and other observations

About 23 hours for the poison to kick in. 24 hours to the feast. What is really being stated here is that you can do only ONE EXPERIMENT. There is no time to do one experiment and then follow that up with another that depends on what happened with the first one. You dish out the wine to the prisoners and 23 hours later, you need to know which bottle is poisoned. That's it. No time for multiple time-sequential experiments.

1000 bottles. 1 poisoned. How many bottles do we need to open? Only 999, since the last one is the poisoned bottle if none of the 999 kill.

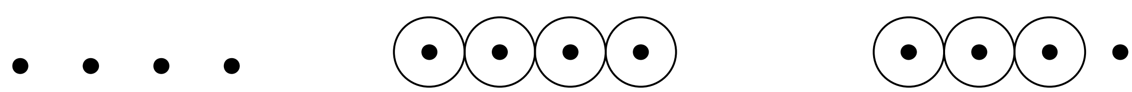
What are we trying to minimize? The number of prisoners who must participate in the experiment. If we were trying to minimize the number of prisoners who die, then the solution would be different - we could give a sip from a unique bottle to a unique prisoners, and then after 23 hours either one or none of the prisoners die. If one dies, the bottle he took a sip from is poisoned. If non died, then the one bottle that was not opened was poisoned (remember only 999 out of the 1000 bottles were opened).

How much wine is required to kill a prisoner? One "sip", I say, but mathematics deals much better with extreme cases than real life. So when I say a "sip" I mean a very very tiny amount ... a molecule.

A prisoners can drink from more than one bottle? This is a pivotal observation. Yes, there is no reason he cannot. And more importantly, he dies if ANY of the bottles he samples is poisoned.

4 bottles, 1 poisoned

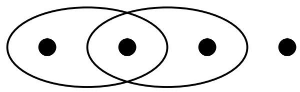
Just to get our head around the problem, lets start with a simple case. This is a technique I find useful to understand the characteristics of the solution, and is universally applicable to many problems. Say there are only 4 bottles. And we know that one is poisoned. One obvious way to solve it is get 4 prisoners to sip from the 4 bottles. One will die. And we know which is the poisoned bottle. But like we discussed earlier, we only need three prisoners since we KNOW that exactly one bottle is poisoned. This is shown in Figure 1.



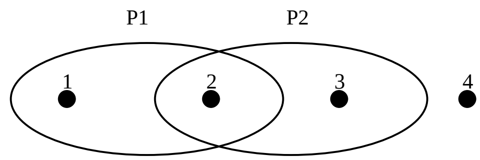
1a. 4 bottles of which one is poisoned 1b. A sip from each bottle to each 1c. One bottle may be left unserved

The dots in Figure 1 are bottles and the circles, or "regions" as I like to call them (because circles have a rigid definition unnecessary for our purposes), enclose a prisoner's tasting choices. To prove to ourselves that there IS A BETTER way to solve the problem and there is a worse way, let's think a little bit further to see if we can do with fewer than prisoners. We use the observation that one prisoner can taste multiple bottles. In other words, the "regions" in Figure 1 expand their selection of wines.

Figure 2 shows how just two regions can suffice. We still leave out one bottle, as always. We have 3 bottles remaining. We use two prisoners and have one taste the first and second bottle of wine, and the second prisoner taste the second and third bottle of wine. That way if the first prisoner dies and the second survives we know it must be the first bottle. If the first prisoner survives and the second dies it must be the third bottle. If both die it must be the second bottle and if none die it must be the fourth (untasted) bottle.



2a. Two prisoners enough



2b. Two prisoners enough
Experiment:
- P1 gets a sip of wine from 1 and 2
- P2 gets a sip of wine from 2 and 3

Analysis:
- P1 dies, P2 lives => 1 is poisoned
- P2 dies, p1 lives => 3 is poisoned
- P1 and P2 die => 2 is poisoned
- P1 and P2 survive => 4 is poisoned

Observations:
- At most 2 prisoners die
- Maybe none die
- 3 bottles need opening
- Most importantly, each bottle is in a uniquely identifiable region: intersection, union, complement of existing regions.

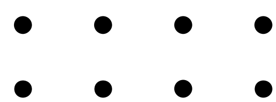
These observations are probably the most important and revealing in getting to the solution. First thing this shows is there is a BETTER WAY to solve the problem than the simple one prisoner - one bottle approach. Secondly, it shows how to draw the regions. The purpose of the regions and its defining characteristic. The purpose of the regions is to divide up the total space (of 4 dots in this case) in such a way that each bottle or dot can be uniquely identified as a combination, intersection, complement of the existing regions. In other words, look at the dots and notice that it shares its unique sub-region with no other dots. There are never two dots in the smallest sub-region. In English, it translates to, every dot has a compound wall made out of the region-walls, that completely encloses it and only it. So no dot has to share accommodation with any other dot. This leads to the characteristic that will help us define what constitutes a region and more importantly, how to draw a region. The region is trying to divide up the existing regions in the most efficient way with the goal of giving each dot its own personal space. The most efficient way it turns out is to contribute to the personal compound wall for as many existing dots that share accommodation, as possible. And the way to do that is to divide EACH existing region into two halves, thereby affecting the most dots until each dot has its own personal space, a personal compound wall, a personal sub-region.

Now lets experiment with more than 4 as the next step to getting all the way to 1000...or even more. You can pick 5, 6 etc. Going with the idea of dividing regions into 2, I will go up to 8 (twice as many as 4).

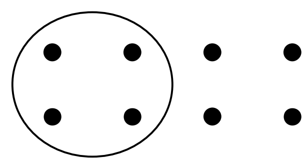
8 bottles, 1 poisoned

Lets start with the 8 bottles. I arrange them in two rows of 4 after some experimentation with drawing regions. You can try any other arrangement that works. The main idea here is we want to be able to draw regions easily. As the number of bottles go up, "drawing" regions on a 2-D piece of paper or even 3-D space is going to get harder...but by then we plan to see the pattern emerge, and then we can give up on "drawing" stuff.

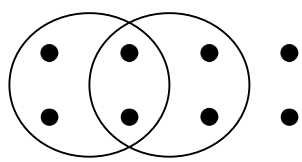
There are 8 bottles. That is one region in itself, even before drawing any region. This is shown in figure 3a. Each dot shares this region with 7 other dots. Then we divide this existing region in to two, as shown in Figure 3b. Notice that there are 4 dots in the region we drew and 4 outside the region we drew. Therefore there are 2 regions now. Each dot shares its region with 3 other dots. We next draw a region which gives some more privacy to each dot. We divide EACH existing region into two equal pieces. This new region is shown in Figure 3c. This region brings to life 4 sub-regions. Each dot now shares space with only 1 other dot. And therefore one more region which nicely divides up EACH existing region should do the job of providing a unique residence for each dot. That is precisely what the region added in Figure 3d does. Notice it neatly slices all the 4 sub-regions, each of which had 2 dots. Now each dot has its own unique space, what I originally called its own compound wall.



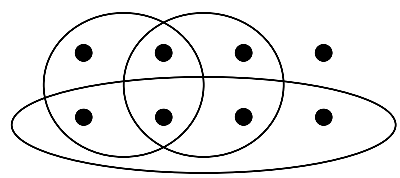
3a. The case of 8 bottles



3b. Two regions (equal sizes)



3c. Divide existing regions evenly



3d. Divide existing regions evenly

Linking this back to the original puzzle, we notice that we need three prisoners, the three ovals/circles/regions we "actually" drew. Lets say all prisoners die. There is only one dot/bottle, which falls into the three prisoners/regions. That is the second dot in the second row. So each dot, now has a unique "address" in terms of regions it falls into. In other words, each bottle has a unique prisoner combination. We simply use that prisoner combination to figure out the bottle, after 23 hours. It works like this:

The first bottle in the first row is poisoned if only the first prisoner dies
The second bottle in the first row is poisoned if only the first and second prisoner die.
The third bottle in the first row is poisoned if only the second prisoner dies.
The fourth bottle in the first row is poisoned if no prisoner dies.
The first bottle in the second row is poisoned if only the first and third prisoner die.
The second bottle in the second row is poisoned if all three prisoners die.
The third bottle in the first second is poisoned if only the second and third prisoner die.
The fourth bottle in the first second is poisoned if only the third prisoner dies.

This should be enough hints to finally solve the problem from any sized set or any number of total bottles. After you have given that a good think proceed to the next section.

Solution

One poisoned bottle

One simple way to think about this problem would be to have a 1000 prisoners sip from the 1000 bottles and the one who dies, drank from the poisoned bottle. A slight improvement is to have 999 prisoners drink from 999 bottles and leave a small chance that no one dies (when the untouched bottle is the poisoned one). But the previous section tells us that there is a distinct advantage to having one prisoner drink from multiple bottles. How then, do we decide who drink from which, and how many drink? We notice from the previous section that the first prisoner drinks from half the total bottles. He corresponds to the first region/circle that is drawn to divide the existing region into two. The next prisoner samples wine from another set, which is also the same size (half the total bottles), but half of it matches what the first prisoner drank and half is from the set the first prisoner did not drink from. Each new prisoner drinks from the same number of bottles. As we saw with 4 bottles, 2 sets/prisoners were required. With 8 bottles, 3 sets/prisoners were required. With 16, 4 would be required, with 32, 5 and so on. With 1024 bottles 10 prisoners would be required. Each of them will sample from 512 bottles. With 1000 bottles, 10 prisoners would still be required. Each of them will at most sample from 512 bottles.

P = ceil(log2(B - 1)), where B is the number of bottles and P is number of prisoners. Ceil is a function that rounds up the number to the next integer (i.e. if the log2(B-1) comes to say 9.81, ceil bumps that up to 10, since we cannot have 9.81 usable prisoners)

At most P prisoners will die and at minimum none will die.

A more elegant solution for the one poisoned bottle case

What we are doing effectively by drawing regions around the dots, is giving each dot a unique "address", a unique "word", using the regions as a smaller set of symbols whose combination provides us the address or word, just as, 10 symbols 0, 1, 2, 3 ... 9 combine to give us a much bigger set of numbers or the 26 letters of the alphabet A,B, C...Z combine to give thousands of words. Each new region we add provides a new level of detail to the dot's address. For example, before drawing region 2, say a dot under consideration falls into region 1. Upon drawing region 2, we have added detail to the dot's address as in "The dot now falls into the region 1 and not in region 2". Since we increase the detail of a dot's "address" one region or symbol at a time, and the only information we add is whether the dot falls in or out of the new region, it starts to look like a growing binary number. So if we simply assign a unique address to each dot or a unique binary identifier to each bottle then the positions where the 1's appear implies that prisoner will taste that wine. For example, if the 8 bottles are numbered 000, 001, 010, 011, 100, 101, 110, 111. Then prisoner 1 will get a sip from bottles marked 001, 011, 101 and 111 (bottles which have a 1 in the first/lowest position). Prisoner 2 will get a sip from bottles marked 010, 011, 110 and 111 (bottles which have a 1 in the 2nd position). Prisoner 3 will get a sip from bottles marked 100, 101, 110 and 111. Say prisoner's 2 and 3 die but 1 lives, then the poisoned bottle is the one marked 110 !! Why not 111? Well, because if 111 were poisoned prisoner 1 would have died too, since he got a sip from that bottle. Why not 100? Because if that were poisoned, then Prisoner 2 would not have died. Only way both Prisoner 1 and 2 would die is if the bottle from which they and ONLY they drank is poisoned.

More than one poisoned bottle?

I still do not have a solution for that yet. At least nothing more interesting than the default one bottle, one prisoner approach.